MTH 303
Real analysis **Homework 7**

1. Show that each of the following sequences of functions converge pointwise on the given domain. Also, find the limit function.
   (a) $f_n(x) = x^n$ for $x \in [0,1]$.
   (b) $f_n(x) = \frac{x}{n}$ for $x \in \mathbb{R}$.
   (c) $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ for $x \in \mathbb{R}$.
   (d) $f_n(x) = n^2 x(1-x^2)^n$ for $x \in [0,1]$.
   (e) $f_n(x) = \begin{cases} 1, & -n \le x \le n \\ 0, & \text{otherwise} \end{cases}$, for $x \in \mathbb{R}$.

2. Check for uniform convergence of sequences described in Question 1.

3. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

4. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, then prove that $\{f_n g_n\}$ converges uniformly on E.

5. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

6. Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$$
   (a) converges for each $x > 0$,
   (b) converges uniformly on any interval of the form $[a,b]$ with $a > 0$.

7. Show that the series
$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$
converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

8. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, consider
$$f_n(x) = \frac{x}{1+nx^2}.$$
Show that $\{f_n\}$ converges uniformly to a function f, and that the equation
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
holds true if $x \neq 0$, but false if $x = 0$.

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that
$$\lim_{n\to\infty} f_n(x_n) = f(x)$$
for every sequence $\{x\_n\}$ in E such that $x_n \to x$ and $x \in E$. Is the converse of this true?

10. For a real-valued continuous function f on $\mathbb{R}$, define $f_n(x) = f(nx)$, $n \in \mathbb{N}$. Assume that the sequence $\{f_n\}$ is equicontinuous on $[0,1]$, then prove that f is constant.

11. For $n \in \mathbb{N}$ and $x \in [0,1]$, consider
$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}.$$
Show that
    (a) $\{f_n\}$ is uniformly bounded on $[0,1]$.
    (b) No subsequence of $\{f_n\}$ can converge uniformly on $[0,1]$.
    (c) $\{f_n\}$ is not equicontinuous on $[0,1]$.

12. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann integrable on $[a,b]$. Define
$$F_n(x) = \int_a^x f_n(t)dt, \quad a \le x \le b.$$
Show that
    (a) Show that $\{F_n\}$ is equicontinuous on $[a,b]$.
    (b) Show that $\{F_n\}$ has a subsequence which is uniformly convergent on $[a,b]$.

13. Show that there exists a sequence of polynomial P_n with $P_n(0) = 0$ such that $P_n(x) \to |x|$ uniformly on $[-1,1]$.

14. If f is a continuous function $[0,1]$ and if
$$\int_0^1 f(x)x^n dx = 0, \quad \text{for each} \quad n \in \mathbb{N},$$
show that $f(x) = 0$ on $[0,1]$.